Monoid of Generalized Hypersubstitutions for Algebraic Systems
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Abstract

In [2], the concept of hypersubstitutions for universal algebras was extended to algebraic systems. The purpose of this paper, we will present the concept of generalized hypersubstitution for algebraic systems which was extended from concept of generalized hypersubstitution for universal algebras (see [1]).

Keywords: algebraic system, term, quantifier free formula, generalized hypersubstitution.

1 Introduction

An algebraic system of type \((\tau, \tau')\) is a triple \(\mathcal{A} := (A; (f^A_i)_{i \in I}, (\gamma^A_j)_{j \in J})\) consisting of a non-empty set \(A\), a sequence \((f^A_i)_{i \in I}\) of operations on \(A\) indexed by the index set \(I\), where \(f^A_i : A^{n_i} \to A\) is \(n_i\)-ary for \(i \in I\) and a sequence \((\gamma^A_j)_{j \in J}\) of relations on \(A\) indexed by the index set \(J\), where \(\gamma^A_j \subseteq A^{n_j}\) is an \(n_j\)-ary relation for \(j \in J\). The pair \((\tau, \tau')\) with \(\tau = (n_i)_{i \in I}, \tau' = (n_j)_{j \in J}\) of sequences of integers \(n_i, n_j \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}\), is called the type of the algebraic system \(\mathcal{A}\) (see [5]).

To classify algebraic systems into subclasses by logical sentences we need a language corresponding to the equational for universal algebras. For universal algebras one needs terms and pairs of terms, i.e. equations. For algebraic systems we need formulas. In this paper, we restrict us to classes of algebraic systems which can be described by quantifier free formulas. To define terms and quantifier free formulas we need variables, operation symbols, logical connectives, and relational symbols.

Let \(1 \leq n \in \mathbb{N}^+\), let \(X_n = \{x_1, x_2, \ldots, x_n\}\) be a finite set of variables, and let \(X := \bigcup_{1 \leq n} X_n = \{x_1, \ldots, x_n, \ldots\}\) be countably infinite. Then the set \(W_r(X_n)\) of all \(n\)-ary terms of type \(\tau\) is defined in the usual way by the following conditions:

(i) Every \(x_i \in X_n\) is an \(n\)-ary term of type \(\tau\).

(ii) If \(t_1, \ldots, t_m\) are \(n\)-ary terms of type \(\tau\) and if \(f_i\) is an \(n_i\)-ary operation symbol of type \(\tau\), then \(f_i(t_1, \ldots, t_{n_i})\) is an \(n\)-ary term of type \(\tau\).
Let \( W_\tau(X) := \bigcup_{n \geq 1} W_\tau(X_n) \) be the set of all terms of type \( \tau \).

To define quantifier free formulas of type \((\tau, \tau')\) we need the logical connectives \( \neg \) (for negation), \( \lor \) (for disjunction) and the equation symbol \( \equiv \).

**Definition 1.1.** Let \( n \in \mathbb{N}^+ \). An \( n \)-ary quantifier free formula (of type \((\tau, \tau')\)) (for short, formula) is defined in the following inductive way:

(i) If \( t_1, t_2 \) are \( n \)-ary terms of type \( \tau \), then the equation \( t_1 \equiv t_2 \) is an \( n \)-ary quantifier free formula of type \((\tau, \tau')\).

(ii) If \( j \in \mathcal{J} \) and \( t_1, \ldots, t_{n_j} \) are \( n \)-ary terms of type \( \tau \), then \( \gamma_j(t_1, \ldots, t_{n_j}) \) is an \( n \)-ary quantifier free formula of type \((\tau, \tau')\).

(iii) If \( F \) is an \( n \)-ary quantifier free formula of type \((\tau, \tau')\), then \( \neg F \) is an \( n \)-ary quantifier free formula of type \((\tau, \tau')\).

(iv) If \( F_1 \) and \( F_2 \) are \( n \)-ary quantifier free formulas of type \((\tau, \tau')\), then \( F_1 \lor F_2 \) is an \( n \)-ary quantifier free formula of type \((\tau, \tau')\).

Let \( \mathcal{F}_{(\tau, \tau')} (X_n) \) be the set of all \( n \)-ary quantifier free formulas of type \((\tau, \tau')\) and let \( \mathcal{F}_{(\tau, \tau')} (X) := \bigcup_{n \geq 1} \mathcal{F}_{(\tau, \tau')} (X_n) \) be the set of all quantifier free formulas of type \((\tau, \tau')\).

## 2 Generalized Superposition of Terms and Formulas

Let \( n \in \mathbb{N}^+ \). We define first the concept of a generalized superposition of terms (see [1]). The operation

\[
S^n : W_\tau(X) \times (W_\tau(X))^n \to W_\tau(X)
\]

by the following steps:

(i) If \( t = x_i; 1 \leq i \leq n \), then \( S^n(x_i, t_1, \ldots, t_n) := t_i \).

(ii) If \( t = x_i; n < i \in \mathbb{N}^+ \), then \( S^n(x_i, t_1, \ldots, t_n) := x_i \).

(iii) If \( t = f_i(s_{i1}, \ldots, s_{in}) \), then \( S^n(f_i(s_{i1}, \ldots, s_{in}), t_1, \ldots, t_n) := f_i(S^n(s_{i1}, t_1, \ldots, t_n), \ldots, S^n(s_{in}, t_1, \ldots, t_n)) \)

supposed that \( S^n(s_k, t_1, \ldots, t_n) \) are already defined for \( 1 \leq k \leq n_i \).

Now, we want to extend this generalize superposition to formulas. If we substitute variables occurring in a formulas by terms, we obtain a new formula.

**Definition 2.1.** The operation

\[
R^n : W_\tau(X) \cup \mathcal{F}_{(\tau, \tau')} (X) \times (W_\tau(X))^n \to W_\tau(X) \cup \mathcal{F}_{(\tau, \tau')} (X)
\]

where \( n \in \mathbb{N}^+ \) are defined by the following inductive steps:

Let \( t_1, \ldots, t_n \in W_\tau(X) \) and \( S^n \) be the generalized superposition of terms.

(i) If \( t \in W_\tau(X) \), then we defined \( R^n(t, t_1, \ldots, t_n) := S^n(t, t_1, \ldots, t_n) \).
(ii) if \( s_1, s_2 \in W_r(X) \), then \( R^n(s_1 \approx s_2, t_1, \ldots, t_n) := S^n(s_1, t_1, \ldots, t_n) \approx S^n(s_2, t_1, \ldots, t_n) \).

(iii) if \( j \in J \) and \( s_{n_j} \in W_r(X) \), then
\[
R^n(\gamma_j(s_1, \ldots, s_{n_j}), t_1, \ldots, t_n) := \gamma_j(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_j}, t_1, \ldots, t_n)).
\]

(iv) if \( F \in \mathcal{F}_{(r,r')} (X) \) and supposed that \( R^n(F, t_1, \ldots, t_n) \) is already defined, then \( R^n(\neg F, t_1, \ldots, t_n) := \neg R^n(F, t_1, \ldots, t_n) \).

(v) if \( F_1, F_2 \in \mathcal{F}_{(r,r')} (X) \) and supposed that \( R^n(F_l, t_1, \ldots, t_n) \) is already defined; \( l \in \{1, 2\} \), then
\[
R^n(F_1 \lor F_2, t_1, \ldots, t_n) := R^n(F_1, t_1, \ldots, t_n) \lor R^n(F_2, t_1, \ldots, t_n).
\]

**Theorem 2.2.** Let \( \beta \in W_r(X) \cup \mathcal{F}_{(r,r')} (X) \). The operation \( R^n \) satisfies:

1. \( R^n(R^p(\beta, t_1, \ldots, t_p), s_1, \ldots, s_n) = R^p(\beta, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n)) \) whenever \( t_1, \ldots, t_p \in W_r(X), s_1, \ldots, s_n \in W_r(X) \).
2. \( R^n(\beta, x_1, \ldots, x_n) = \beta \).

**Proof.** For \( \beta = t \in W_r(X) \), we will give a proof of (FC1) by induction on the complexity of term \( t \).

(i) if \( t = x_i; 1 \leq i \leq n \leq p \), then
\[
R^n(R^p(x_i, t_1, \ldots, t_p), s_1, \ldots, s_n)
= R^n(S^p(x_i, t_1, \ldots, t_p), s_1, \ldots, s_n)
= S^n(t_1, s_1, \ldots, s_n)
= S^p(x_i, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_p, s_1, \ldots, s_n))
= R^n(x_i, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n)).
\]

(ii) if \( t = x_i; i > p \geq n \), then
\[
R^n(R^p(x_i, t_1, \ldots, t_p), s_1, \ldots, s_n)
= R^n(x_i, s_1, \ldots, s_n)
= S^p(x_i, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_p, s_1, \ldots, s_n))
= R^n(x_i, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n)).
\]

(iii) if \( t = f_i(q_1, \ldots, q_{n_i}) \) and assume that \( S^n(S^p(q_i, t_1, \ldots, t_p), s_1, \ldots, s_n) = S^p(q_i, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_p, s_1, \ldots, s_n)) ; 1 \leq i \leq n_i \), then
\[
R^n(R^p(f_i(q_1, \ldots, q_{n_i}), t_1, \ldots, t_p), s_1, \ldots, s_n)
= R^n(S^p(f_i(q_1, \ldots, q_{n_i}), t_1, \ldots, t_p), s_1, \ldots, s_n)
= R^n(f_i(S^p(q_1, t_1, \ldots, t_p), s_1, \ldots, s_n), \ldots, S^p(q_{n_i}, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_p, s_1, \ldots, s_n)))
= f_i(S^p(q_1, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_p, s_1, \ldots, s_n)), \ldots, S^p(q_{n_i}, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_p, s_1, \ldots, s_n)))
= S^p(f_i(q_1, \ldots, q_{n_i}), S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_p, s_1, \ldots, s_n))
= R^n(f_i(q_1, \ldots, q_{n_i}), R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n)).
\]

For \( \beta = F \in \mathcal{F}_{(r,r')} (X) \), we will give a proof of (FC1) by induction on the complexity of formula \( F \).
(i) If \( F \) has the from \( q_1 \approx q_2 \), then
\[
R^n(R^p(q_1 \approx q_2, t_1, \ldots, t_p), s_1, \ldots, s_n)
= R^n(S^p(q_1, t_1, \ldots, t_p) \approx S^p(q_2, t_1, \ldots, t_p), s_1, \ldots, s_n)
= S^n(S^p(q_1, t_1, \ldots, t_p), s_1, \ldots, s_n) \approx S^n(S^p(q_2, t_1, \ldots, t_p), s_1, \ldots, s_n)
= S^p(q_1, s^n(t_1, s_1, \ldots, s_n), \ldots, s^n(t_p, s_1, \ldots, s_n)) \approx
S^p(q_2, s^n(t_1, s_1, \ldots, s_n), \ldots, s^n(t_p, s_1, \ldots, s_n)).
\]
By Definition 2.1(ii), this is equal to the formula
\[
R^n(q_1 \approx q_2, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_p, s_1, \ldots, s_n))
= R^n(q_1 \approx q_2, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n)).
\]

(ii) If \( F \) has the from \( \gamma_j(q_1, \ldots, q_{n_j}) \), then
\[
R^n(R^p(\gamma_j(q_1, \ldots, q_{n_j}), t_1, \ldots, t_p), s_1, \ldots, s_n)
= R^n(\gamma_j(S^p(q_1, t_1, \ldots, t_p), \ldots, S^p(q_{n_j}, t_1, \ldots, t_p)), s_1, \ldots, s_n)
= \gamma_j(S^n(S^p(q_1, t_1, \ldots, t_p), s_1, \ldots, s_n), \ldots,
S^n(S^p(q_{n_j}, t_1, \ldots, t_p), s_1, \ldots, s_n))
= \gamma_j(S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_p, s_1, \ldots, s_n)), \ldots
S^n(q_{n_j}, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_p, s_1, \ldots, s_n)))
= R^n(\gamma_j(q_1, \ldots, q_{n_j}), S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_p, s_1, \ldots, s_n))
= R^n(\gamma_j(q_1, \ldots, q_{n_j}), R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n)).
\]

(iii) If \( F \) has the from \( \neg F \) and assume that \( R^n(R^p(F, t_1, \ldots, t_p), s_1, \ldots, s_n) \)
\[
= R^n(\neg F, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n)), then
R^n(R^p(\neg F, t_1, \ldots, t_p), s_1, \ldots, s_n)
= R^n(\neg R^p(F, t_1, \ldots, t_p), s_1, \ldots, s_n)
= \neg R^n(R^p(F, t_1, \ldots, t_p), s_1, \ldots, s_n)
= \neg R^n(F, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n))
= R^n(\neg F, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n)).
\]

(iv) If \( F \) has the from \( F_1 \lor F_2 \) and assume that \( R^n(R^p(F_i, t_1, \ldots, t_p), s_1, \ldots, s_n) \)
\[
= R^n(F_1, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n)); l \in \{1, 2\}, then
R^n(R^p(F_1 \lor F_2, t_1, \ldots, t_p), s_1, \ldots, s_n)
= R^n(R^p(F_1, t_1, \ldots, t_p) \lor R^p(F_2, t_1, \ldots, t_p), s_1, \ldots, s_n)
= R^n(R^p(F_1, t_1, \ldots, t_p), s_1, \ldots, s_n) \lor R^n(R^p(F_2, t_1, \ldots, t_p), s_1, \ldots, s_n)
= R^n(F_1, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n))
\lor R^n(F_2, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n))
= R^n(F_1 \lor F_2, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_p, s_1, \ldots, s_n)).
\]

This finishes the proof of (FC1).

The proof of (FC3), we will proceed in a similar way considering the completely of term \( t \).

(i) If \( t = x_i; 1 \leq i \leq n \), then
\[
R^n(x_i, x_1, \ldots, x_n) = S^n(x_i, x_1, \ldots, x_n)
= x_i.
\]

(ii) If \( t = x_i; n < i \), then
\[
R^n(x_i, x_1, \ldots, x_n) = S^n(x_i, x_1, \ldots, x_n)
= x_i.
\]

(iii) If \( t = f_i(s_1, \ldots, s_{n_i}) \) and assume that \( R^n(s_1, x_1, \ldots, x_n) = s_1; 1 \leq l \leq n_i \), then
\[
R^n(f_i(s_1, \ldots, s_{n_i}), x_1, \ldots, x_n) = S^n(f_i(s_1, \ldots, s_{n_i}), x_1, \ldots, x_n)
= f_i(S^n(s_1, x_1, \ldots, x_n), \ldots, S^n(s_{n_i}, x_1, \ldots, x_n))
= f_i(s_1, \ldots, s_{n_i}).
\]

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The proof of (FC3), we will proceed in a similar way considering the completely of formula $F$.

(i) If $F$ has the from $s_1 \approx s_2$, then 
$$R^n(s_1 \approx s_2, x_1, \ldots, x_n) = S^n(s_1, x_1, \ldots, x_n) \approx S^n(s_2, x_1, \ldots, x_n) = s_1 \approx s_2.$$

(ii) If $F$ has the from $\gamma_j(s_1, \ldots, s_n_l)$, then 
$$R^n(\gamma_j(s_1, \ldots, s_{n_j}), x_1, \ldots, x_n) = \gamma_j(S^n(s_1, x_1, \ldots, x_n), \ldots, S^n(s_{n_j}, x_1, \ldots, x_n)) = \gamma_j(s_1, \ldots, s_{n_j}).$$

(iii) If $F$ has the from $\neg F$ and assume that $R^n(F, x_1, \ldots, x_n) = F$, then 
$$R^n(\neg F, x_1, \ldots, x_n) = \neg R^n(F, x_1, \ldots, x_n) = \neg F.$$

(iv) If $F$ has the from $F_1 \lor F_2$ and assume that $R^n(F_1, x_1, \ldots, x_n) = F_i; i \in \{1, 2\}$, then 
$$R^n(F_1 \lor F_2, x_1, \ldots, x_n) = R^n(F_1, x_1, \ldots, x_n) \lor R^n(F_2, x_1, \ldots, x_n) = F_1 \lor F_2.$$

Example 2.3. : We consider the type $(\tau, \tau') = ((2), (2))$, i.e. there are one binary operation symbol $f$ and one binary relational symbol $\gamma$. We calculate 
$$R^2(\gamma(f(x_1, f(x_2, x_3)), f(f(x_1, x_4), x_5)), f(x_1, x_2), x_3)$$
$$= \gamma(S^2(f(x_1, f(x_2, x_3)), f(x_1, x_2), x_3), S^2(f(f(x_1, x_4), x_5), f(x_1, x_2), x_3))$$
$$= \gamma(f(S^2(f(x_1, f(x_1, x_2), x_3), S^2(f(f(x_1, x_4), f(x_1, x_2), x_3)), f(f(x_1, x_4), f(x_1, x_2), x_3), S^2(f(x_1, x_2), x_3), S^2(x_5, f(x_1, x_2), x_3))$$
$$= \gamma(f(f(x_1, x_2), f(x_3, x_3)), f(f(f(x_1, x_2), f(x_3, x_3), f(f(x_1, x_2), x_3)).$$
This means, we obtain the result if we substitute in $f(x_1, f(x_2, x_3))$ and $f(f(x_1, x_4), x_5)$ for $x_1$ the term $f(x_1, x_2)$ for $x_2$ the term $x_3$ and let $x_3, x_4, x_5$ untouched.

3 Monoid of Generalized Hypersubstitutions for Algebraic Systems

A generalized hypersubstitution for algebraic system of type $(\tau, \tau')$ is mapping 
$$\sigma : \{f_i \mid i \in I\} \cup \{\gamma_j \mid j \in J\} \rightarrow W_\tau(X) \cup F_{(\tau, \tau')}(X)$$
which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions for algebraic systems by $Hyp^G(\tau, \tau')$.

Every generalized hypersubstitution for algebraic system $\sigma$ can be extended to a mapping 
$$\tilde{\sigma} : W_\tau(X) \cup F_{(\tau, \tau')}(X) \rightarrow W_\tau(X) \cup F_{(\tau, \tau')}(X)$$
inductively defined as follows:

(i) $\tilde{\sigma}[x] := x$ for any variable $x \in X$,

(ii) $\tilde{\sigma}[f_i(s_1, \ldots, s_{n_i})] := S^n(\sigma(f_i), \tilde{\sigma}[s_1], \ldots, \tilde{\sigma}[s_{n_i}])$ for $i \in I$ and $s_1, \ldots, s_{n_i} \in W_\tau(X)$,

(iii) $\tilde{\sigma}[s_1 \approx s_2] := \tilde{\sigma}[s_1] \approx \tilde{\sigma}[s_2]$ for $s_1, s_2 \in W_\tau(X)$,
(v) $\tilde{\sigma}[\gamma_j(s_1, \ldots, s_n)] := R^n_j(\sigma(\gamma_j), \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n])$ for $j \in J$ and $s_1, \ldots, s_n \in W_\tau(X)$,

(vi) $\tilde{\sigma}[\neg F] := \neg \tilde{\sigma}[F]$ for $F \in \mathcal{F}_{(\tau, \tau')}(X)$,

(vii) $\tilde{\sigma}[F_1 \lor F_2] := \tilde{\sigma}[F_1] \lor \tilde{\sigma}[F_2]$ for $F_1, F_2 \in \mathcal{F}_{(\tau, \tau')}(X)$.

Then $\tilde{\sigma}$ is called the extension of a generalized hypersubstitution for algebraic system $\sigma$.

**Example 3.1.** Let $\tau = (2, 1)$ and $\tau' = (2)$ be the type, i.e. we have one binary operation symbol, one unary operation symbol and one binary relational symbol, say $f, g$ and $\gamma$, respectively. Let $\sigma : \{f, g\} \cup \{\gamma\} \rightarrow W_{(2, 1)}(X) \cup \mathcal{F}_{(2, 1), (2)}(X)$ where $\sigma(f) = f(g(x_2), x_5), \sigma(g) = f(x_4, x_1)$ and $\sigma(\gamma) = x_1 \approx f(x_1, x_3)$. Then we have

\[
\tilde{\sigma}[\gamma(f(x_1, x_3), g(x_2))] = R^2(\sigma(\gamma), \hat{\sigma}[f(x_1, x_3)], \hat{\sigma}[g(x_2)]) = R^2(x_1 \approx f(x_1, x_3), S^2(\sigma(f), \hat{\sigma}[x_1], \hat{\sigma}[x_5]), S^1(\sigma(g), \hat{\sigma}[x_2])) = R^2(x_1 \approx f(x_1, x_3), S^2(f(g(x_2), x_5), x_1, x_5), S^1(f(x_4, x_1), x_2)) = R^2(x_1 \approx f(x_1, x_3), f(g(x_5), x_5), f(x_4, x_2)) = S^2(x_1, f(g(x_5), x_5), f(x_4, x_2)) \approx S^2(f(x_1, x_3), f(g(x_5), x_5), f(x_4, x_2)) = f(g(x_5), x_5) \approx f(f(g(x_5), x_5), x_3).
\]

Then we define a binary operation $\circ_G$ on $Hyp^G(\tau, \tau')$ by $\sigma_1 \circ_G \sigma_2 := \tilde{\sigma}_1 \circ \tilde{\sigma}_2$ where $\circ$ denotes the usual composition of mapping and $\sigma_1, \sigma_2 \in Hyp^G(\tau, \tau')$.

**Proposition 3.2.** Let $\sigma \in Hyp^G(\tau, \tau')$, let $n \in \mathbb{N}^+$ and let $t_1, \ldots, t_n \in W_\tau(X)$. Then

\[
\tilde{\sigma}[R^n(\sigma(\beta), t_1, \ldots, t_n)] = R^n(\tilde{\sigma}[\beta], \tilde{\sigma}[t_1], \ldots, \tilde{\sigma}[t_n])
\]

for any $\beta \in W_\tau(X) \cup \mathcal{F}_{(\tau, \tau')}(X)$.

**Proof.** For $\beta = t \in W_\tau(X)$, we will give a proof by induction on the complexity of the definition of term $t$.

If $t = x_i; 1 \leq i \leq n$, then

\[
R^n(\tilde{\sigma}[x_i], \tilde{\sigma}[t_1], \ldots, \tilde{\sigma}[t_n]) = S^n(x_i, \tilde{\sigma}[t_1], \ldots, \tilde{\sigma}[t_n]) = \tilde{\sigma}[x_i] = \tilde{\sigma}[S^n(x_i, t_1, \ldots, t_n)] = \tilde{\sigma}[R^n(x_i, t_1, \ldots, t_n)].
\]

If $t = x_i; i > n$, then

\[
R^n(\tilde{\sigma}[x_i], \tilde{\sigma}[t_1], \ldots, \tilde{\sigma}[t_n]) = S^n(x_i, \tilde{\sigma}[t_1], \ldots, \tilde{\sigma}[t_n]) = x_i = \tilde{\sigma}[x_i] = \tilde{\sigma}[S^n(x_i, t_1, \ldots, t_n)] = \tilde{\sigma}[R^n(x_i, t_1, \ldots, t_n)].
\]

Now, we assume that $t = f_i(s_1, \ldots, s_{n_i})$ and that for $s_1, \ldots, s_{n_i}$ our proposition is satisfied. Then

\[
R^n(\tilde{\sigma}[f_i(s_1, \ldots, s_{n_i})], \tilde{\sigma}[t_1], \ldots, \tilde{\sigma}[t_n]) = R^n(\tilde{\sigma}[f_i], \tilde{\sigma}[s_1], \ldots, \tilde{\sigma}[s_{n_i}], \tilde{\sigma}[t_1], \ldots, \tilde{\sigma}[t_n]) = R^n(\tilde{\sigma}[f_i], S^n(\tilde{\sigma}[s_1], \tilde{\sigma}[t_1], \ldots, \tilde{\sigma}[t_n]), \ldots, \tilde{\sigma}[s_{n_i}], \tilde{\sigma}[t_1], \ldots, \tilde{\sigma}[t_n]) = R^n(\tilde{\sigma}[f_i], S^n(s_1, t_1, \ldots, t_n), \ldots, \tilde{\sigma}[S^n(s_{n_i}, t_1, \ldots, t_n)]) = \tilde{\sigma}[f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))] = \tilde{\sigma}[R^n(f_i, s_1, \ldots, s_{n_i}), t_1, \ldots, t_n].
\]

For $\beta = F \in \mathcal{F}_{(\tau, \tau')}(X)$, we will give a proof by induction on the complexity of the definition of formula $F$. 

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(i) If $F$ has the form $s_1 \approx s_2$, then
\[ \hat{\sigma}[R^n(s_1 \approx s_2, t_1, \ldots, t_n)] = \hat{\sigma}[S^n(s_1, t_1, \ldots, t_n) = S^n(s_2, t_1, \ldots, t_n)] = S^n(\hat{\sigma}[s_1], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = S^n(\hat{\sigma}[s_2], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = R^n(\hat{\sigma}[s_1] \approx \hat{\sigma}[s_2], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = R^n(\hat{\sigma}[s_1 \approx s_2], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]). \]

(ii) If $F$ has the form $\gamma_j(s_1, \ldots, s_{n_j})$, then
\[ \hat{\sigma}[R^n(\gamma_j(s_1, \ldots, s_{n_j}), t_1, \ldots, t_n)] = \hat{\sigma}[\gamma_j(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_j}, t_1, \ldots, t_n))] = R^n(\gamma_j(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_j}, t_1, \ldots, t_n))) = R^n(R^n(\sigma(\gamma_j), \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_{n_j}]), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = R^n(\hat{\sigma}[\gamma_j(s_1, \ldots, s_{n_j})], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]). \]

(iii) If $F$ has the form $\neg F$ and assume that $\hat{\sigma}[R^n(F, t_1, \ldots, t_n)] = R^n(\hat{\sigma}[F], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$, then
\[ \hat{\sigma}[R^n(\neg F, t_1, \ldots, t_n)] = \hat{\sigma}[\neg R^n(F, t_1, \ldots, t_n)] = \neg(\hat{\sigma}[R^n(F, t_1, \ldots, t_n)]) = \neg(R^n(\hat{\sigma}[F], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])) = R^n(\hat{\sigma}[\neg F], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = R^n(\hat{\sigma}[\neg F], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]). \]

(iv) If $F$ has the form $F_1 \lor F_2$ and assume that $\hat{\sigma}[R^n(F_1, t_1, \ldots, t_n)] = R^n(\hat{\sigma}[F_1], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]); l \in \{1, 2\},$ then
\[ \hat{\sigma}[R^n(F_1 \lor F_2, t_1, \ldots, t_n)] = \hat{\sigma}[R^n(F_1, t_1, \ldots, t_n) \lor R^n(F_2, t_1, \ldots, t_n)] = \hat{\sigma}[R^n(F_1, t_1, \ldots, t_n)] \lor \hat{\sigma}[R^n(F_2, t_1, \ldots, t_n)] = R^n(\hat{\sigma}[F_1], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \lor R^n(\hat{\sigma}[F_2], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = R^n(\hat{\sigma}[F_1 \lor F_2], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]). \]

Lemma 3.3. For any $\sigma_1, \sigma_2 \in Hyp^G(\tau, \tau')$, we have
\[ (\sigma_1 \circ_G \sigma_2) = \hat{\sigma}_1 \circ \hat{\sigma}_2. \]

Proof. Let $t \in W_\tau(X)$, we will give a proof by induction on the complexity of the definition of the term $t$.

(i) If $t = x$, then $(\sigma_1 \circ_G \sigma_2)[x] = x = \hat{\sigma}_1[\hat{\sigma}_2[x]] = (\hat{\sigma}_1 \circ \hat{\sigma}_2)[x].$

(ii) If $t = f_i(t_1, \ldots, t_{n_i})$ for any $i \in I$ and assume that $(\sigma_1 \circ_G \sigma_2)[t_k] = \hat{\sigma}_1 \circ \hat{\sigma}_2[t_k]$ for every $k \in \{1, \ldots, n_i\}$.
Then
\[ (\sigma_1 \circ_G \sigma_2)[f_i(t_1, \ldots, t_{n_i})] = R^n((\sigma_1 \circ_G \sigma_2)(f_i), (\sigma_1 \circ_G \sigma_2)[t_1], \ldots, (\sigma_1 \circ_G \sigma_2)[t_{n_i}]) = R^n((\hat{\sigma}_1 \circ \hat{\sigma}_2)(f_i), (\hat{\sigma}_1 \circ \hat{\sigma}_2)[t_1], \ldots, (\hat{\sigma}_1 \circ \hat{\sigma}_2)[t_{n_i}]) = R^n(\hat{\sigma}_1, \hat{\sigma}_2[f_i(t_1, \ldots, t_{n_i})]) = \hat{\sigma}_1 R^n(\sigma_2[f_i(t_1, \ldots, t_{n_i})]) \text{ (by Proposition 3.2)} = \hat{\sigma}_1 \circ \hat{\sigma}_2[f_i(t_1, \ldots, t_{n_i})] = (\hat{\sigma}_1 \circ \hat{\sigma}_2)[f_i(t_1, \ldots, t_{n_i})]. \]
For $F \in F_{(r,r')}(X_n)$, we will give a proof by induction on the complexity of a formula $F$.

(i) If $F$ has the form $t_1 \approx t_2$, then $(\sigma_1 \circ \sigma_2)[t_1 \approx t_2]$
\begin{align*}
&= (\sigma_1 \circ \sigma_2)[t_1] \approx (\sigma_1 \circ \sigma_2)[t_2] \\
&= (\sigma_1 \circ \sigma_2)[t_1] \\
&= \sigma_1[\sigma_2[t_1] \approx \sigma_1[\sigma_2[t_2]] \\
&= \sigma_1[\sigma_2[t_1] \approx \sigma_2[t_2]] \\
&= \sigma_1[\sigma_2[t_1 \approx t_2]] \\
&= (\sigma_1 \circ \sigma_2)[t_1 \approx t_2].
\end{align*}

(ii) If $F$ has the form $\gamma_j(t_1, \ldots, t_{n_j})$, then $(\sigma_1 \circ \sigma_2)[\gamma_j(t_1, \ldots, t_{n_j})]$
\begin{align*}
&= R^{n_j}((\sigma_1 \circ \sigma_2)(\gamma_j), (\sigma_1 \circ \sigma_2)[t_1, \ldots, (\sigma_1 \circ \sigma_2)[t_{n_j}])) \\
&= R^{n_j}((\sigma_1 \circ \sigma_2)(\gamma_j), \sigma_1[\sigma_2[t_1], \ldots, (\sigma_1 \circ \sigma_2)[t_{n_j}])] \\
&= \sigma_1[R^{n_j}(\gamma_j, \sigma_2[t_1], \ldots, \sigma_2[t_{n_j}])] \\
&= \sigma_1[\gamma_j(t_1, \ldots, t_{n_j})] \\
&= (\sigma_1 \circ \sigma_2)[\gamma_j(t_1, \ldots, t_{n_j})].
\end{align*}
\hspace{1cm} (by Proposition 3.2)

(iii) If $F$ has the form $\neg F$ and assume that $(\sigma_1 \circ \sigma_2)[F] = (\sigma_1 \circ \sigma_2)[F]$, then
\begin{align*}
(\sigma_1 \circ \sigma_2)[\neg F] \\
&= \neg((\sigma_1 \circ \sigma_2)[F]) \\
&= \neg((\sigma_1 \circ \sigma_2)[F]) \\
&= (\sigma_1[\neg F]) \\
&= \sigma_1[\neg F] \\
&= (\sigma_1 \circ \sigma_2)[\neg F].
\end{align*}

(iv) If $F$ has the form $F_1 \lor F_2$ and assume that $(\sigma_1 \circ \sigma_2)[F_i] = (\sigma_1 \circ \sigma_2)[F_i]$; $i \in \{1, 2\}$, then
\begin{align*}
(\sigma_1 \circ \sigma_2)[F_1 \lor F_2] \\
&= (\sigma_1 \circ \sigma_2)[F_1] \lor (\sigma_1 \circ \sigma_2)[F_2] \\
&= (\sigma_1 \circ \sigma_2)[F_1] \lor (\sigma_1 \circ \sigma_2)[F_2] \\
&= \sigma_1[\sigma_2[F_1] \lor \sigma_1[\sigma_2[F_2]] \\
&= \sigma_1[\sigma_2[F_1] \lor \sigma_2[F_2]] \\
&= \sigma_1[\sigma_2[F_1] \lor \sigma_2[F_2]] \\
&= (\sigma_1 \circ \sigma_2)[F_1 \lor F_2].
\end{align*}

Let $\sigma_{id}$ be the generalized hypersubstitution for algebraic systems mapping the operation symbols $f_i$ to the terms $f_i(x_1, \ldots, x_{n_i})$ for all $i \in I$, and the relational symbols $\gamma_j$ to the formulas $\gamma_j(x_1, \ldots, x_{n_j})$ for all $j \in J$.

Lemma 3.4. For any term $t \in W_+(X)$ and formula $F \in F_{(r,r')}(X)$, we have $\sigma_{id}[t] = t$ and $\sigma_{id}[F] = F$.

Proof. Let $n \in \mathbb{N}^+$ and $t \in W_+(X)$, we will give a proof by induction on the definition of term $t$.

(i) If $t = x_i$ with $i \in 1 \leq i \leq n$, then $\sigma_{id}[x_i] = x_i$.

(ii) If $t = f_i(s_1, \ldots, s_{n_i})$ for $i \in I$ and assume that $\sigma_{id}[s_l] = s_l; 1 \leq l \leq n_i$ , then $\sigma_{id}[f_i(s_1, \ldots, s_{n_i})]$.  

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For \( F \in \mathcal{F}(\tau, \tau')(X) \), we will give a proof by induction on the definition of formula \( F \).

(i) If \( F \) has the form \( s_1 \approx s_2 \), then \( \bar{\sigma}_{id}[s_1] \approx \bar{\sigma}_{id}[s_2] \)
\[ \begin{align*}
= & \ \bar{\sigma}_{id}[s_1] \approx \bar{\sigma}_{id}[s_2] \\
= & \ s_1 \approx s_2
\end{align*} \]

(ii) If \( F \) has the form \( \gamma_j(s_1, \ldots, s_{n_j}) \), then \( \bar{\sigma}_{id}[\gamma_j(s_1, \ldots, s_{n_j})] \)
\[ \begin{align*}
= & \ R^{\gamma_j}(\bar{\sigma}_{id}(\gamma_j), \bar{\sigma}_{id}[s_1], \ldots, \bar{\sigma}_{id}[s_{n_j}]) \\
= & \ R^{\gamma_j}(\gamma_j(x_1, \ldots, x_{n_j}), s_1, \ldots, s_{n_j}) \\
= & \ \bar{\gamma}_j(S^{\gamma_j}(x_1, s_1, \ldots, s_{n_j}), \ldots, S^{\gamma_j}(x_{n_j}, s_1, \ldots, s_{n_j})) \\
= & \ \gamma_j(s_1, \ldots, s_{n_j}).
\end{align*} \]

(iii) If \( F \) has the form \( \neg F \) and assume that \( \bar{\sigma}_{id}[F] = F \), then \( \bar{\sigma}_{id}[-F] \)
\[ \begin{align*}
= & \ \neg \bar{\sigma}_{id}[F] \\
= & \ \neg F.
\end{align*} \]

(iv) If \( F \) has the form \( F_1 \lor F_2 \) and assume that \( \bar{\sigma}_{id}[F_l] = F_l; l \in \{1, 2\} \), then \( \bar{\sigma}_{id}[F_1 \lor F_2] \)
\[ \begin{align*}
= & \ \bar{\sigma}_{id}[F_1] \lor \bar{\sigma}_{id}[F_2] \\
= & \ F_1 \lor F_2.
\end{align*} \]

Theorem 3.5. The set \( Hyp^G(\tau, \tau') \) forms the monoid
\[ Hyp^G(\tau, \tau') := (Hyp^G(\tau, \tau'); \circ_G, \sigma_{id}). \]

Proof. Using Lemma 3.3 and using the fact that \( \circ \) is associative, it can be shown that \( \circ_G \) is associative. In fact, we have
\[ \begin{align*}
\sigma_1 \circ_G (\sigma_2 \circ_G \sigma_3) \\
= & \ \bar{\sigma}_1 \circ (\bar{\sigma}_2 \circ \sigma_3) \\
= & \ ((\bar{\sigma}_1 \circ \bar{\sigma}_2) \circ \sigma_3) \\
= & \ (\sigma_1 \circ \sigma_2) \circ \sigma_3 \\
= & \ \sigma_1 \circ \sigma_2 \circ \sigma_3.
\end{align*} \] (by Lemma 3.3)

Lemma 3.4 shows that \( \sigma_{id} \) is an identity element with respect to \( \circ_G \). In fact, if \( \beta \in \{f_i|i \in I\} \cup \{\gamma_j|j \in J\} \), then \( (\sigma_{id} \circ_G \sigma)(\beta) \)
\[ \begin{align*}
= & \ (\sigma_{id} \circ_G \sigma)(\beta) \\
= & \ (\bar{\sigma}_{id} \circ \sigma)(\beta) \\
= & \ \bar{\sigma}_{id}[\sigma(\beta)] \\
= & \ \sigma(\beta)
\end{align*} \] (by Lemma 3.4).

If \( i \in I \), then \( (\sigma \circ_G \sigma_{id})(f_i) \)
\[ \begin{align*}
= & \ (\bar{\sigma} \circ_G \bar{\sigma}_{id})(f_i) \\
= & \ \bar{\sigma}_i[\sigma_{id}(f_i)] \\
= & \ \bar{\sigma}_i[f_i(x_1, \ldots, x_{n_i})] \\
= & \ R^{\sigma_i}(\sigma(f_i), \bar{\sigma}[x_1], \ldots, \bar{\sigma}[x_{n_i}]) \\
= & \ R^{\sigma_i}(\sigma(f_i), x_1, \ldots, x_{n_i}) \\
= & \ \sigma(f_i)
\end{align*} \] (by Theorem 2.2 (FC3)).
If $j \in J$, then $(\sigma \circ_G \sigma_{id})(\gamma_j)$
\[= (\tilde{\sigma} \circ_G \sigma_{id})(\gamma_j)\]
\[= \tilde{\sigma}[\sigma_{id}(\gamma_j)]\]
\[= \tilde{\sigma}[\gamma_i(x_1, \ldots, x_{n_j})] \quad \text{(by defined } \sigma_{id}(\gamma_j)\text{)}\]
\[= R^{n_j}(\sigma(\gamma_j), \tilde{\sigma}[x_1], \ldots, \tilde{\sigma}[x_{n_j}])\]
\[= R^{n_j}(\sigma(\gamma_j), x_1, \ldots, x_{n_j})\]
\[= \sigma(\gamma_j) \quad \text{(by Theorem 2.2 (FC3))}.\]

Therefore $\sigma \circ_G \sigma_{id} = \sigma_{id} \circ_G \sigma = \sigma$.

References


